Exact correlators of two-matrix models

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Abstract

We compute exact solutions of two-matrix models, i.e. detailed genus by genus expressions for the correlation functions of these theories, calculated without any approximation. We distinguish between two types of models, the unconstrained and the constrained ones. Unconstrained two-matrix models represent perturbations of $c = 1$ string theory, while the constrained ones correspond to topological field theories coupled to topological gravity. Among the latter we treat, in particular detail, the ones based on the KdV and on the Boussinesq hierarchies.

1. Introduction

Matrix models represent sums over discretizations of Riemann surfaces, possibly with some additional interactions. They are believed to provide a (discrete) description of two-dimensional gravity coupled to matter. One-matrix models have been widely investigated, but their content is rather poor. The structure of multi-matrix models is much richer but not yet known as carefully as for one-matrix models (see [1–14]).

In this paper, we concentrate on two-matrix models with bilinear coupling and show how to find *exact solutions*. By exact solutions we mean detailed genus by genus expressions for the correlation functions of these theories, calculated without any approximation. In

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particular, we do not limit ourselves to exhibiting recursion relations which allow one to compute correlators, but develop techniques to explicitly solve them.

The idea at the basis of our treatment of two-matrix models, outlined in [15], is to transform the initial functional integral problem into a discrete integrable linear system subject to some constraints (the coupling conditions). We end up in this way with the discrete Toda lattice hierarchy. The latter underlies all our calculations: our aim is to compute the correlation functions (CFs) of each model, which in turn may be expressed in terms of the integrable flows of the Toda hierarchy, subject to the coupling conditions. This is the general setting for *unconstrained* two-matrix models (which simply denote two-matrix models defined by specific potentials without any further conditions).

We will also consider other models, obtained by suitably constraining the previous (unconstrained) models. Their correlators can be expressed either in terms of the flows of a reduced differential hierarchy or in terms of suitably renormalized flows of the discrete Toda hierarchy. In the process of solving these models we find a new way of extracting integrable differential hierarchies from the Toda lattice hierarchies.

Unconstrained two-matrix models describe various perturbations of $c = 1$ string theory at the self-dual point. Constrained models correspond to well-known topological field theories coupled to topological gravity.

The paper is organized as follows. In Section 2 we collect the results obtained in previous papers which will be necessary in the following. In Section 3 we discuss and calculate CFs of *unconstrained models*. Section 4 is a short summary of how to obtain differential hierarchies from the Toda lattice flows and reduced hierarchies via hamiltonian reduction. In Section 5, from the reduced hierarchies we construct a series of models, named reduced models, which have a topological field theory meaning: we show in particular how to compute all genus correlators. The reduced models turn out to be embedded into the constrained two-matrix models, which are studied and solved in Section 6.

2. General properties of two-matrix models

The model of two Hermitian $N \times N$ matrices M_1 and M_2, is introduced in terms of the partition function

$$Z_N(t, c) = \int dM_1 \, dM_2 e^{\operatorname{tr} U}, \quad U = V_1 + V_2 + g M_1 M_2 \tag{2.1}$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} \bar{t}_{\alpha,r} M_\alpha^r, \quad \alpha = 1, 2, \tag{2.2}$$

where p_α are finite numbers. These potentials define the model. We denote by $\mathcal{M}_{p_1, p_2}$ the corresponding two-matrix model.

We are interested in computing correlation functions (CFs) of the operators

$$\tau_k = \operatorname{tr} M_1^k, \quad \sigma_k = \operatorname{tr} M_2^k, \quad \forall k, \quad \chi = \operatorname{tr}(M_1 M_2).$$

For this reason we complete the above model by replacing (2.2) with the more general potentials

$$V_\alpha = \sum_{r=1}^{\infty} t_{\alpha,r} M_\alpha^r, \quad \alpha = 1, 2, \tag{2.3}$$

where $t_{\alpha,r} \equiv \bar{t}_{\alpha,r}$ for $r \le p_\alpha$. The CFs are defined by

$$\langle \tau_{r_1} \ldots \tau_{r_n} \sigma_{s_1} \ldots \sigma_{s_m} \chi^l \rangle = \frac{\partial^{n+m+l}}{\partial t_{1,r_1} \ldots \partial t_{1,r_n} \partial t_{2,s_1} \ldots \partial t_{2,s_m} \partial g^l} \ln Z_N(t, g), \tag{2.4}$$

where, in the right-hand side, all the $t_{\alpha,r}$ except $\bar{t}_{\alpha,r}$ are set to zero.

In other words we have embedded the original couplings $\bar{t}_{\alpha,r}$ into two infinite sets of couplings. Therefore we have two types of couplings. The first type consists of those couplings (the barred ones) that define the model: they represent the true *dynamical* parameters of the theory; they are kept non-vanishing throughout the calculations. The second type encompasses the remaining couplings, which are introduced only for computational purposes and are set to zero in formulae like (2.4). In terms of ordinary field theory the former are analogous to the interaction couplings, while the latter correspond to external sources (coupled to composite operators). From now on we will not make any formal distinction between them. Case by case we will specify which are the interaction couplings and which are the external ones. Finally, it is sometime convenient to consider N on the same footing as the couplings and to set $t_{1,0} \equiv t_{2,0} \equiv N$.

The path integral (2.1) is an ordinary integral in the matrix entries and it is certainly well defined as long as a negative coupling $\bar{t}_{\alpha,r}$ with highest even r guarantees that the measure is square-integrable and decreases more than polynomially at infinity. For the time being let us suppose that this is so. Later on we will extend our problem to a larger coupling space.

We briefly recall the ordinary procedure to calculate the partition function. It consists of three steps [17–19]:

(i) One integrates out the angular part so that only the integrations over the eigenvalues are left.

(ii) One introduces the orthogonal monic polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \qquad \eta_n(\lambda_2) = \lambda_2^n + \text{lower powers},$$

which satisfy the orthogonality relations

$$\int d\lambda_1 \ d\lambda_2 \xi_n(\lambda_1) e^{\mu(\lambda_1,\lambda_2)} \eta_m(\lambda_2) = h_n(t, c) \delta_{nm}, \tag{2.5}$$

where

$$\mu(\lambda_1, \lambda_2) = V_1(\lambda_1) + V_2(\lambda_2) + c\lambda_1\lambda_2.$$

(iii) Using the orthogonality relation (2.5) and the properties of the Vandermonde determinants, one can easily calculate the partition function

$$Z_N(t, c) = \text{const } N! \prod_{i=0}^{N-1} h_i. \tag{2.6}$$

2.1. From path integral to integrable systems

From (2.6) we see that knowing the partition function means knowing the coefficients $h_n(t, c)$.

The crucial point, from our point of view, is that the information concerning the latter can be encoded in (1) a suitable linear system subject to certain (2) coupling conditions, together with (3) relations that allow us to reconstruct Z_N. But before we pass to these three elements we need some convenient notations. For any matrix M, we define

$$\mathcal{M} = H^{-1} M H, \qquad H_{ij} = h_i \delta_{ij}, \qquad \bar{M}_{ij} = M_{ji}, \qquad M_l(j) \equiv M_{j,j-l}.$$

As usual we introduce the natural gradation

$$\deg[E_{ij}] = j - i, \quad \text{where } (E_{i,j})_{k,l} = \delta_{i,k} \delta_{j,l}$$

and, for any given matrix M, if all its non-zero elements have degrees in the interval $[a, b]$, then we will simply write: $M \in [a, b]$. Moreover M_+ will denote the upper triangular part of M (including the main diagonal), while $M_- = M - M_+$. We will write

$$\text{Tr}(M) = \sum_{i=0}^{N-1} M_{ii}.$$

The latter operation will be referred to as taking the finite trace.

Next we pass from the basis of orthogonal polynomials to the basis of orthogonal functions

$$\Psi_n(\lambda_1) = e^{V_1(\lambda_1)} \xi_n(\lambda_1), \qquad \Phi_n(\lambda_2) = e^{V_2(\lambda_2)} \eta_n(\lambda_2).$$

The orthogonality relation (2.5) becomes

$$\int d\lambda_1 \, d\lambda_2 \Psi_n(\lambda_1) e^{c\lambda_1\lambda_2} \Phi_m(\lambda_2) = \delta_{nm} h_n(t, c). \tag{2.7}$$

We will denote by Ψ the semi-infinite column vector $(\Psi_0, \Psi_1, \Psi_2, \ldots,)^t$ and by Φ the vector $(\Phi_0, \Phi_1, \Phi_2, \ldots,)^t$.

Then we introduce the following Q-type matrices

$$\int d\lambda_1 \, d\lambda_2 \Psi_n(\lambda_1) \lambda_\alpha e^{c\lambda_1\lambda_2} \Phi_m(\lambda_2) \equiv Q_{nm}(\alpha) h_m = \bar{Q}_{mn}(\alpha) h_n, \quad \alpha = 1, 2. \tag{2.8}$$

Both $Q(1)$ and $\bar{Q}(2)$ are Jacobi matrices: their pure upper triangular part is $I_+ = \sum_i E_{i,i+1}$.

Besides the above Q matrices, we will need two P-type matrices, defined by

$$\int d\lambda_1 \, d\lambda_2 \left(\frac{\partial}{\partial \lambda_1} \Psi_n(\lambda_1) \right) e^{c\lambda_1\lambda_2} \Phi_m(\lambda_2) \equiv P_{nm}(1) h_m, \tag{2.9}$$

$$\int d\lambda_1 \, d\lambda_2 \Psi_n(\lambda_1) e^{c\lambda_1\lambda_2} \left(\frac{\partial}{\partial \lambda_2} \Phi_m(\lambda_2) \right) \equiv P_{mn}(2) h_n. \tag{2.10}$$

For later use we also introduce

$$\int d\lambda_1 \ d\lambda_2 \left(\frac{\partial}{\partial \lambda_1} \xi_n(\lambda_1) \right) \exp\{V_1(\lambda_1) + V_2(\lambda_2) + c\lambda_1\lambda_2\}\eta_m(\lambda_2) \equiv P^\circ_{nm}(1)h_m,$$

$$(2.11)$$

$$\int d\lambda_1 \ d\lambda_2 \xi_n(\lambda_1) \exp\{V_1(\lambda_1) + V_2(\lambda_2) + c\lambda_1\lambda_2\} \left(\frac{\partial}{\partial \lambda_2} \eta_m(\lambda_2) \right) \equiv P^\circ_{mn}(2)h_n.$$

$$(2.12)$$

Let us come now to the three elements announced above.

(1) *Coupling conditions.* The two matrices (2.8) we introduced above are not completely independent. More precisely both $Q(\alpha)$'s can be expressed in terms of only one of them and one matrix P. Expressing the trivial fact that the integral of the total derivative of the integrand in Eq. (2.7) with respect to λ_1 and λ_2 vanishes, we can easily derive the constraints or *coupling conditions*:

$$P(1) + cQ(2) = 0, \qquad cQ(1) + \bar{P}(2) = 0. \tag{2.13}$$

From the coupling conditions it follows at once that, if we set to zero the external couplings,

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \qquad \alpha = 1, 2,$$

where

$$m_1 = p_2 - 1, \quad m_2 = 1 \quad \text{and} \quad n_1 = 1, \quad n_2 = p_1 - 1,$$

where p_α, $\alpha = 1, 2$ is the highest order of the interacting part of the potential V_α (see above).

(2) *The associated linear systems.* The derivation of the linear systems associated to our matrix model is very simple. We take the derivatives of Eq. (2.7) with respect to the time parameters $t_{\alpha,r}$, and use Eq. (2.8). We get in this way the time evolution of Ψ, or *discrete linear system I*:

$$Q(1)\Psi(\lambda_1) = \lambda_1 \Psi(\lambda_1),$$
$$\frac{\partial}{\partial t_{1,k}} \Psi(\lambda_1) = Q^k(1)_+ \Psi(\lambda_1),$$
$$\frac{\partial}{\partial t_{2,k}} \Psi(\lambda_1) = -Q^k(2)_- \Psi(\lambda_1), \tag{2.14}$$
$$\frac{\partial}{\partial \lambda} \Psi(\lambda_1) = P(1)\Psi(\lambda_1).$$

The corresponding consistency conditions are

$$[Q(1), \ P(1)] = 1, \tag{2.15a}$$
$$\frac{\partial}{\partial t_{\alpha,k}} Q(1) = [Q(1), \ Q^k(\alpha)_-], \qquad \alpha = 1, 2. \tag{2.15b}$$

In a similar way we can get the time evolution of Φ via a *discrete linear system II*, whose consistency conditions are

$$[\bar{Q}(2), \quad P(2)] = 1, \tag{2.16a}$$

$$\frac{\partial}{\partial t_{\alpha,k}} Q(2) = [Q^k(\alpha)_+, \quad Q(2)]. \tag{2.16b}$$

We recall that one can write down flows for $P(1)$ and $P(2)$, but they will not be used in this paper.

(3) *Reconstruction formulae.* The third element announced above is the link between the quantities that appear in the linear system and in the coupling conditions with the original partition function. We have

$$\frac{\partial}{\partial_{\alpha,r}} \ln Z_N(t, c) = \text{Tr}(Q^r(\alpha)), \quad \alpha = 1, 2. \tag{2.17}$$

It is evident that, by using Eqs. (2.15b) and (2.16b) above we can express all the derivatives of Z_N in terms of the elements of the Q matrices. For example

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} \ln Z_N(t, c) = (Q^r(\alpha))_{N,N-1}, \quad \alpha = 1, 2 \tag{2.18}$$

and so on. We recall that the derivatives of $F(N, t, c) = \ln Z_N(t, c)$ are nothing but the correlation functions of the model.

We can summarize the content of this section in the following:

Proposition 2.1. *The correlators (2.4) can be expressed in terms of the entries of the matrices $Q(1)$ and $Q(2)$ via Eq. (2.17) and the like. In turn, these matrices must satisfy the coupling conditions (2.13) and the consistency conditions (2.15a)–(2.16b).*

Knowing all the derivatives with respect to the coupling parameters we can reconstruct the partition function up to an overall integration constant (depending only on N). The reconstructed free energy F will be a power series in the external couplings.

This theorem was proven in [15]. It is a rigorous result when, for example, highest negative even couplings guarantee that the measure in (2.5) is square-integrable and decreases more than polynomially at infinity. But for generic values of the couplings the above derivations are merely heuristic.

However we notice that the consistency and coupling conditions make sense for any value of the couplings, and also when the couplings are infinite in number. In the latter case Eqs. (2.15b) and (2.16b) form nothing but a very well-known discrete integrable hierarchy, the Toda lattice hierarchy, [20] (see also [21]). From these considerations it is clearly very convenient to refer to the integrable system formulation rather than the original path integral formulation of our problem. This allows us not only to extend our problem to a larger region of the parameter space, but also to make full use of integrability. Therefore we shift from the original problem to the new formulation:

We call (unconstrained) two-matrix models as all those models obtained by specifying a partition of the couplings between internal and external couplings. Each such model is based on the Toda lattice hierarchy and characterized by specific coupling conditions.

Statement of the problem. Solve the integrable Toda lattice hierarchy subject to the coupling conditions specific of a particular unconstrained model and compute the correlators as functions of the internal couplings via the relations

$$\langle \tau_r \rangle = \text{Tr}(Q(1)^r), \qquad \langle \sigma_s \rangle = \text{Tr}(Q(2)^s) \tag{2.19}$$

and the like.

Once all the correlators are known, one can reconstruct [1] the free energy F by means of

$$\frac{\partial}{\partial t_{1,r}} F(N, t, c) = \langle \tau_r \rangle, \qquad \frac{\partial}{\partial t_{2,r}} F(N, t, c) = \langle \sigma_r \rangle. \tag{2.20}$$

F will be a formal power series in the infinite set of external couplings.

Henceforth this will be the setup we refer to.

To end this subsection, we collect a few formulae we will need later on. First, we will be using the following coordinates of the Jacobi matrices:

$$Q(1) = I_+ + \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i,i-l}, \qquad \bar{Q}(2) = I_+ + \sum_i \sum_{l=0}^{m_2} b_l(i) E_{i,i-l}. \tag{2.21}$$

One can immediately see that

$$(Q_+(1))_{ij} = \delta_{j,i+1} + a_0(i)\delta_{i,j}, \qquad (Q_-(2))_{ij} = R(i)\delta_{j,i-1}, \tag{2.22}$$

where $R(i + 1) \equiv h_{i+1}/h_i$. As a consequence of these coordinates, Eq. (2.18) gives in particular the two important relations

$$\frac{\partial^2}{\partial t_{1,1}^2} F(N, t, c) = a_1(N). \tag{2.23}$$

Finally we write down explicitly the $t_{1,1}$- and $t_{2,1}$-flows, which will play a very important role in what follows:

$$\frac{\partial}{\partial t_{1,1}} a_l(j) = a_{l+1}(j+1) - a_{l+1}(j) + a_l(j)(a_0(j) - a_0(j-l)), \tag{2.24a}$$

$$\frac{\partial}{\partial t_{2,1}} a_l(j) = R(j-l+1)a_{l-1}(j) - R(j)a_{l-1}(j-1), \tag{2.24b}$$

$$\frac{\partial}{\partial t_{1,1}} b_l(j) = R(j-l+1)b_{l-1}(j) - R(j)b_{l-1}(j-1), \tag{2.24c}$$

$$\frac{\partial}{\partial t_{2,1}} b_l(j) = b_{l+1}(j+1) - b_{l+1}(j) + b_l(j)(b_0(j) - b_0(j-l)). \tag{2.24d}$$

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[1] Up to a constant depending only on N. There is a way to determine this constant too, see [16], but we will not dwell upon this point here.

2.2. W_∞ constraints

To solve the above stated problem we have to solve the flow equations of the Toda lattice hierarchy subject to the coupling conditions (2.13). There is a way to put together flow equations and coupling conditions that lead to an elegant algebraic structure, the W constraints:

Proposition 2.2. *The partition function of the unconstrained two-matrix models satisfies the following W constraints:*

$$W_n^{[r]} Z_N(t, c) = 0, \qquad \tilde{W}_n^{[r]} Z_N(t, c) = 0, \qquad r \geq 0; \quad n \geq -r, \tag{2.25}$$

where

$$W_n^{[r]} \equiv (-c)^n \mathcal{L}_n^{[r]}(1) - \mathcal{L}_{-n}^{[r+n]}(2), \tag{2.26a}$$

$$\tilde{W}_n^{[r]} \equiv (-c)^n \mathcal{L}_n^{[r]}(2) - \mathcal{L}_{-n}^{[r+n]}(1). \tag{2.26b}$$

The generators $\mathcal{L}_n^{[r]}(1)$ are differential operators involving N and $t_{1,k}$, while $\mathcal{L}_n^{[r]}(2)$ have the same form with $t_{1,k}$ replaced by $t_{2,k}$. One of the remarkable aspects of (2.25) is that the dependence on the coupling c is nicely factorized. The $\mathcal{L}_n^{[r]}(1)$ satisfy the following W_∞ algebra:

$$[\mathcal{L}_n^{[1]}(1), \mathcal{L}_m^{[1]}(1)] = (n - m)\mathcal{L}_{n+m}^{[1]}(1), \tag{2.27a}$$

$$[\mathcal{L}_n^{[2]}(1), \mathcal{L}_m^{[1]}(1)] = (n - 2m)\mathcal{L}_{n+m}^{[2]}(1) + m(m + 1)\mathcal{L}_{n+m}^{[1]}(1), \tag{2.27b}$$

$$[\mathcal{L}_n^{[2]}(1), \mathcal{L}_m^{[2]}(1)] = 2(n - m)\mathcal{L}_{n+m}^{[3]}(1) - (n - m)(n + m + 3)\mathcal{L}_{n+m}^{[2]}(1), \tag{2.27c}$$

and in general

$$[\mathcal{L}_n^{[r]}(1), \mathcal{L}_m^{[s]}(1)] = (sn - rm)\mathcal{L}_{n+m}^{[r+s-1]}(1) + \cdots, \tag{2.28}$$

for $r, s \geq 1$; $n \geq -r, m \geq -s$. Here dots denote lower than $r + s - 1$ rank operators. We notice that this W_∞ algebra is not simple, as it contains a Virasoro subalgebra spanned by the $\mathcal{L}_n^{[1]}(1)$'s. For this reason, it is often called $W_{1+\infty}$ algebra. We also see that once we know these generators and $\mathcal{L}_{-2}^{[2]}(1)$, the remaining ones are produced by the algebra itself.

The algebra of the $\mathcal{L}_n^{[r]}(2)$ is just a copy of the above one, and the algebra satisfied by the $W_n^{[r]}$ and by $\tilde{W}_n^{[r]}$ is isomorphic to both.

The derivation of the W constraints is very simple [15]. It consists of taking the coupling conditions (2.13), multiplying them by powers of $Q(1)$ and $Q(2)$, taking the finite trace and using the flow equations of the Toda lattice hierarchy. This was done in detail in Ref. [15]. There, one can also find explicit expressions of the generators, see also [22].

2.3. Homogeneity and genus expansion

The CFs we compute are genus expanded. The genus expansion is strictly connected with the homogeneity properties of the CFs. As we will see the contribution pertinent to

any genus is a homogeneous function of the couplings (and N) with respect to appropriate degrees assigned to all the involved quantities. Precisely, we assign to the couplings the following degrees:

$$\deg(\ ) \equiv [\ ], \quad [t_{\alpha,k}] = y - y_\alpha k, \quad [N] = y, \quad [c] = y - y_1 - y_2, \tag{2.29}$$

where y, y_1, y_2 are arbitrary constants. Here and in the following N is treated as a coupling $t_{1,0} = t_{2,0}$.

If we rescale the couplings as

$$t_{\alpha,k} \rightarrow \lambda^{[t_{\alpha,k}]} t_{\alpha,k},$$

on the basis of the analysis of Bessis et al. [17], we expect the free energy to scale like

$$F \rightarrow \sum_{h=0}^{\infty} \lambda^{2y(1-h)} F^{(h)}, \tag{2.30}$$

where $F^{(h)}$ is the genus h contribution. In other words

$$[F^{(h)}] = 2y(1 - h). \tag{2.31}$$

The CFs will be expanded accordingly. Such expectation, based on a path integral analysis, remains true in our setup due to the fact that the homogeneity properties carry over to the Toda lattice hierarchy. To this end we have simply to consider a genus expansion for all the coordinate fields that appear in $Q(1)$ and $Q(2)$, see (2.21), (2.22). The Toda lattice hierarchy splits accordingly. In genus 0 the assignments

$$[a_l^{(0)}] = (l + 1)y_1, \qquad [b_l^{(0)}] = (l + 1)y_2, \qquad [R^{(0)}] = y_1 + y_2 \tag{2.32}$$

correspond exactly to the assignments (2.29) and $[F^{(0)}] = 2y$.

It is very common to replace the matrix size N with a continuum variable, say x. This is permitted provided one rescales all the quantities involved according to the above degrees, [16].

3. Correlation functions in (unconstrained) matrix models

We have (at least) three methods to calculate CFs. The first consists of directly solving the W constraints; the second consists of determining from the coupling conditions the explicit form of $Q(1)$ and $Q(2)$ and then using the flows of the discrete Toda lattice hierarchy; the third method is based on passing from the discrete hierarchy to a purely differential one and integrating the flows of the latter. The first method has been shown in a number of examples, [16,22]. Moreover we will see it at work in the constrained models. Therefore we skip it here and pass directly to the second method.

3.1. Solving the coupling conditions: $\mathcal{M}_{2,2}$ model

This method is based on explicitly solving the coupling constraints (2.13). It is then elementary to compute correlators by means of Eq. (2.17) and the lattice Toda flow equations. First we discuss in detail the model $\mathcal{M}_{2,2}$, i.e. the purely Gaussian case. For simplicity we set

$$t_{1,k} \equiv t_k, \qquad t_{2,k} \equiv s_k.$$

Lemma 3.1. *The matrices $Q(1)$ and $Q(2)$ relevant for the model $\mathcal{M}_{2,2}$ are specified, with reference to the coordinates (2.21), (2.22), by*

$$a_0(n) \equiv a_0 = \frac{cs_1 - 2s_2t_1}{4s_2t_2 - c^2}, \qquad a_1(n) = -\frac{2s_2n}{4s_2t_2 - c^2},$$

$$b_0(n) \equiv b_0 = \frac{ct_1 - 2t_2s_1}{4s_2t_2 - c^2}, \qquad b_1(n) = -\frac{2t_2n}{4s_2t_2 - c^2}, \qquad R(n) = \frac{nc}{4s_2t_2 - c^2}.$$

$$(3.1)$$

The remaining coordinates vanish.

Proof. The coupling conditions (2.13) for the model $\mathcal{M}_{2,2}$ are

$$P^\circ(1) + t_1 + 2t_2 Q(1) + cQ(2) = 0,$$
$$\overline{\mathcal{P}}^\circ(2) + s_1 + 2s_2 Q(2) + cQ(1) = 0.$$

Using the fact that $P^\circ_{n,n-1}(i) = n$ for $i = 1, 2$, they can be explicitly written in terms of the coordinates as

$$2t_2 R(n) + cb_1(n) = 0, \qquad ca_1(n) + 2s_2 R(n) = 0,$$
$$t_1 + 2t_2 a_0(n) + cb_0(n) = 0, \qquad s_1 + ca_0(n) + 2s_2 b_0(n) = 0,$$
$$n + 2t_2 a_1(n) + cR(n) = 0, \qquad cR(n) + 2s_2 b_1(n) + n = 0.$$

These equations can be easily solved and give (3.1). □

Proposition 3.2. *The exact one-point and two-point correlators of the model $\mathcal{M}_{2,2}$ are given by*

$$\langle \tau_r \rangle = \sum_{2l=0}^{r} \sum_{k=0}^{l} \frac{r! \, 2^{-k}}{(r - 2l)! k! (l - k)!} \binom{N}{l - k + 1} \left(\frac{2s_2}{c^2 - 4s_2t_2} \right)^{l-k}$$

$$\times \left(\frac{2s_2t_1 - cs_1}{c^2 - 4s_2t_2} \right)^{r-2l}.$$

$$(3.2)$$

Proof. To start with it is convenient to rewrite $Q(1)$ and $Q(2)$ as

$$Q(i) = \alpha_i I_+ + \beta_i I + \gamma_i \epsilon_-, \qquad i = 1, 2,$$

$$(3.3)$$

where:

$$I_+ = \sum_{n=0}^{\infty} E_{n,n+1}, \qquad I = \sum_{i=0}^{\infty} E_{n,n}, \qquad \epsilon_- = \sum_{n=1}^{\infty} n E_{n,n-1},$$

and

$$
\begin{array}{lll}
\alpha_1 = 1, & \beta_1 = a_0, & \gamma_1 = \dfrac{2s_2}{c^2 - 4s_2 t_2}, \\[2mm]
\alpha_2 = -\dfrac{2t_2}{c}, & \beta_2 = b_0, & \gamma_2 = \dfrac{c}{4s_2 t_2 - c^2}.
\end{array}
\tag{3.4}
$$

By means of the formulae

$$[I_+, \epsilon_-] = I, \qquad \mathrm{Tr}\left(\epsilon_-^k I_+^l\right) = \delta_{k,l} \sum_{n=k}^{N-1} \frac{n!}{(n-k)!} = \delta_{n,k}\, k! \binom{N}{k+1}, \tag{3.5}$$

we can now make explicit computations. For example

$$
\begin{aligned}
\mathrm{Tr}\left(Q(1)^r\right) &= \sum_{2l=0}^{r} \binom{r}{2l} \mathrm{Tr}(I_+ + \gamma_1 \epsilon_-)^{2l} \beta_1^{r-2l} \\
&= \sum_{2l=0}^{r} \sum_{k=0}^{l} \frac{r!\, 2^{-k}}{(r-2l)!\, k!\, (l-k)!} \binom{N}{l-k+1} \gamma_1^l \beta_1^{r-2l}.
\end{aligned}
\tag{3.6}
$$

From this formula, using (2.17), we can immediately get Eq. (3.2). In a similar way we can derive $\langle \sigma_r \rangle$.

Finally using the genus expansion (2.30) we can extract the genus by genus correlators. □

Corollary 3.3. *The genus h contributions to the one- and two-point CFs in the model $\mathcal{M}_{2,2}$ are*

$$
\begin{aligned}
\langle \tau_r \rangle_h &= \sum_{2l=0}^{r} \sum_{k=0}^{l} \frac{(-1)^{2h-k} 2^{-k} r!\, \beta_{2h-k}(l-k) N^{l+1-2h}}{(r-2l)!\, k!\, (l-k)!\, (l-k+1)!} \left(\frac{2s_2}{c^2 - 4s_2 t_2}\right)^l \\
&\quad \times \left(\frac{2s_2 t_1 - c s_1}{c^2 - 4s_2 t_2}\right)^{r-2l},
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
&\beta_k(r) = \sum_{1 \le r_1 \le r_2 \le \cdots \le r_k \le r} r_1 r_2 \ldots r_k, \qquad 1 \le k \le r, \\
&\beta_0(r) = 1, \qquad \beta_k(r) = 0, \quad \text{otherwise.}
\end{aligned}
$$

Due to the β factor the sum over l in (3.7) starts at $2h$ and the sum over k ends at $2h$. For the two-point correlators, see Appendix A.

We have given the above proof in some detail since it constitutes a model for all the other more complicated cases. In fact there is nothing new when we consider the $\mathcal{M}_{p,1}$ models. They can be solved exactly in the same way. New features appear in the case of

the $\mathcal{M}_{p_1, p_2}$ models with $p_1, p_2 > 1$ and $p_1 + p_2 > 4$, since the constraints give rise to non-linear algebraic relations for the coordinates. Let us see the simplest possible example of this situation.

3.2. Solving the coupling conditions: $\mathcal{M}_{3,2}$ model

The coupling conditions of the $\mathcal{M}_{3,2}$ model are

$$
\begin{aligned}
&P^\circ(1) + 3t_3 Q(1)^2 + 2t_2 Q(1) + t_1 + cQ(2) = 0, \\
&\bar{P}^\circ(2) + 2s_2 Q(2) + s_1 + cQ(1) = 0.
\end{aligned}
\tag{3.8}
$$

Using the coordinates (2.21) and (2.22) we find that the fields $a_l(n)$, $b_l(n)$, $R(n)$ must satisfy the equations:

$$
\begin{aligned}
&cb_2(n) + 3t_3 R(n) R(n-1) = 0, \\
&2t_2 R(n) + cb_1(n) + 3t_3 R(n)(a_0(n) + a_0(n-1)) = 0, \\
&3t_3(a_0(n)^2 + a_1(n) + a_1(n+1)) + 2t_2 a_0(n) + t_1 + cb_0(n) = 0, \\
&n + 3t_3 a_1(n)(a_0(n) + a_0(n-1)) + 2t_2 a_1(n) + cR(n) = 0, \\
&2s_2 R(n) + ca_1(n) = 0, \\
&2s_2 b_0(n) + s_1 + ca_0(n) = 0, \\
&n + 2s_2 b_1(n) + cR(n) = 0.
\end{aligned}
\tag{3.9}
$$

One easily realizes that the second, fourth, fifth and seventh equations are linearly dependent. Finally one has

$$
a_1(n) = -\frac{2s_2}{c} R(n), \qquad b_0(n) = -\frac{s_1 + ca_0(n)}{2s_2},
$$

$$
b_1(n) = -\frac{n + cR(n)}{2s_2}, \qquad b_2(n) = -\frac{3t_3}{c} R(n) R(n-1),
$$

and the recursion relations:

$$
a_0(n) + a_0(n-1) = -\frac{2t_2}{3t_3} + \frac{c}{6s_2 t_3}\left(c + \frac{n}{R(n)}\right),
\tag{3.10}
$$

$$
R(n+1) + R(n) = \frac{c}{2s_2} a_0(n)^2 + \left(\frac{2ct_2}{6s_2 t_3} - \frac{c^3}{12s_2^2 t_3}\right) a_0(n) - \frac{c^2 s_1}{12s_2^2 t_3} + \frac{ct_1}{6s_2 t_3}.
\tag{3.11}
$$

These recursion relations can be solved exactly in complete generality, although the final formulae may look very cumbersome. However, for our present purposes, it will be sufficient to see the solutions in genus 0. In genus 0 the above equations become:

$$
a_1(n) = -\frac{2s_2}{c} R(n), \qquad b_0(n) = -\frac{s_1 + ca_0(n)}{2s_2},
$$

$$
b_1(n) = -\frac{n + cR(n)}{2s_2}, \qquad b_2(n) = -\frac{3t_3}{c} R(n)^2,
\tag{3.12}
$$

and the recursion relations

$$2a_0(n) = -\frac{2t_2}{3t_3} + \frac{c}{6s_2t_3}\left(c + \frac{n}{R(n)}\right), \tag{3.13}$$

$$2R(n) = \frac{c}{2s_2}a_0(n)^2 + \left(\frac{2ct_2}{6s_2t_3} - \frac{c^3}{12s_2^2t_3}\right)a_0(n) - \frac{c^2s_1}{12s_2^2t_3} + \frac{ct_1}{6s_2t_3}. \tag{3.14}$$

This leads to a cubic equation for a_0. Once this equation is solved with the standard formulae, all the fields are completely determined. Since they are not particularly illuminating, we do not write down here the explicit solutions. We notice however that, once we know them, it is possible to write down immediately an integral expression for the correlation functions. For example, using the same formulae as in the previous subsection, one gets

$$\langle \tau_r \rangle_0 = \sum_{l \geq r/2}^{r} \frac{r!}{((r-l)!)^2(2l-r)!} \int^x dn \bar{a}_0^{2l-r} \bar{a}_1^{r-l}, \tag{3.15}$$

where $\bar{a}_0, \bar{a}_1$ are the solutions of the above algebraic system, and we have promoted n and N to continuum variables and called the latter x. In a similar way we can obtain all the correlators we wish.

As we see from this example, the method is the same as in the $\mathcal{M}_{2,2}$ model, the only additional difficulty being the solution of a third-order algebraic equation. When we come to more complicated $\mathcal{M}_{p_1,p_2}$ models, the method remains the same but we are faced with the problem of solving higher-order algebraic equations.

It has been shown in [16] (see also [23]) that the model $\mathcal{M}_{0,0}$ represents the $c = 1$ string theory at the self-dual point. Any $\mathcal{M}_{p_1,p_2}$ model represents the perturbation of the former by the corresponding tachyonic states. We have shown that these perturbations can, at least in principle, be solved. However, we do not see any point in pushing the analysis further in this direction. We would rather like to concentrate from now on a related interesting problem: *can one obtain from non-Gaussian $\mathcal{M}_{p_1,p_2}$ models 'simple' submodels, in the sense that, for example, the correlators are polynomials of the couplings?* The answer is yes, and the submodels are obtained by imposing constraints in the coordinates of the $\mathcal{M}_{p_1,p_2}$ models. The submodels are called *constrained two-matrix models* and to the analysis of some of them are devoted the next three sections.

4. Differential hierarchies of two-matrix models

One possible characterization of the constrained models is by means of the differential integrable hierarchies they correspond to.

Let us return to Section 2. We saw there that two-matrix models can be represented by means of coupled discrete linear systems, whose consistency conditions give rise to the Toda lattice hierarchy. Here we review the method, used in [15], to transform the discrete linear systems into equivalent differential systems whose consistency conditions are purely differential hierarchies. This is tantamount to separating the N dependence from

the dependence on the couplings. This section is introduced for the sake of completeness: we collect and try to render as plausible as possible the results obtained in [15,24] we will need in the following.

The clue to the construction are the first flows, i.e. the $t_{1,1}$ and $t_{2,1}$ flows. For the sake of simplicity let us consider the *system I* and the flow (2.24a). Let us consider the generic situation in which $Q(1)$ has $m_1 = p_2 - 1$ lower diagonal lines (see the parametrization (2.21)). To begin with we notice that

$$\frac{\partial}{\partial t_{1,1}} \Psi_n = \Psi_{n+1} + a_0(n) \Psi_n \tag{4.1}$$

and adopt for any function $f(t)$ the convention $f' \equiv \partial f / \partial t_{1,1} \equiv \partial f$. We can rewrite

$$\Psi_n = \hat{B}_n \Psi_{n+1}, \tag{4.2}$$

where

$$\hat{B}_n \equiv \frac{1}{\partial - a_0(n)} = \partial^{-1} \sum_{l=0}^{\infty} (a_0(n) \partial^{-1})^l. \tag{4.3}$$

In so doing we implicitly understand that the framework in which we operate is that of the pseudodifferential calculus, see for example [25].

It is now an easy exercise to prove that the discrete spectral equation

$$Q(1) \Psi(\lambda_1) = \lambda_1 \Psi(\lambda_1)$$

is transformed into the pseudodifferential one

$$L_n(1) \Psi_n = \lambda_1 \Psi_n, \tag{4.4}$$

where

$$L_n(1) = \partial + \sum_{l=1}^{m_1} a_l(n) \hat{B}_{n-l} \hat{B}_{n-l+1} \dots \hat{B}_{n-1}$$

$$= \partial + \sum_{l=1}^{m_1} a_l(n) \frac{1}{\partial - a_0(n-l)} \cdot \frac{1}{\partial - a_0(n-l+1)} \dots \frac{1}{\partial - a_0(n-1)}. \tag{4.5}$$

Proceeding in the same way for the other equations of *system I* we obtain the new system in differential form

$$L_n(1) \Psi_n = \lambda_1 \Psi_n,$$

$$\frac{\partial}{\partial t_{1,r}} \Psi_n = (L_n^r(1))_+ \Psi_n, \tag{4.6}$$

The subscript $+$ appended to a pseudodifferential operator represents the purely differential part of it. The subscript $-$ represents the complementary part.

Let us come now to the n dependence of the above equations. The operator $L_n(1)$ in (4.5) depends on the coordinates of different lattice points. To deal with this complication, we introduce m_1 "fields" $S_1, \ldots, S_{m_1}$, related to the "field" a_0 in the following way:

$$S_i(n) \equiv a_0(n - i). \tag{4.7}$$

Then we can rewrite $L_n(1)$ as

$$L_n(1) = \partial + \sum_{l=1}^{m_1} a_l(n) \frac{1}{\partial - S_l(n)} \cdot \frac{1}{\partial - S_{l-1}(n)} \cdots \frac{1}{\partial - S_1(n)} \tag{4.8}$$

with the result that $L_n(1)$ is expressed in terms of fields evaluated at the same lattice point. Of course the fields S_i are not independent. However, we will consider these fields as completely independent from one another in all the intermediate steps of our calculations and only eventually impose the condition (4.7).

To further simplify the notation we will consider henceforth the lattice label n on the same footing as the couplings and write

$$a_i(n, \ldots) \equiv a_i(n)(\ldots), \qquad S_i(n, \ldots) \equiv S_i(n)(\ldots),$$

where the dots denote the dependence on $t_{1,k}$, $t_{2,k}$ and g. So the expression of $L_1(n)$ gets further simplified to

$$L = \partial + \sum_{l=1}^{m} a_l \frac{1}{\partial - S_l} \cdot \frac{1}{\partial - S_{l-1}} \cdots \frac{1}{\partial - S_1}, \tag{4.9}$$

where, for simplicity, we have dropped the label (1) too. This simplified form is the one we constantly refer to throughout this and the following section.

A similar treatment can be applied to the second linear system as well. Therefore *the information concerning matrix models can be stored in two differential linear systems + the first flow equations* (2.24a)–(2.24d). The former determine the dependence on the couplings, while the latter fix the dependence on N. Therefore what we have accomplished so far is the separation of the dependence on N from the dependence on the couplings.

From now on we refer to the consistency conditions

$$\frac{\partial}{\partial t_r} L = [(L^r)_+, L], \tag{4.10}$$

where $t_r \equiv t_{1,r}$. (4.10) are integrable hierarchies, [26,27], which are classified by the number $2m$ of fields. The pseudodifferential operator L in (4.9) is the relevant Lax operator.

We can easily locate these hierarchies in a well-known framework. In fact L is nothing but a particular realization of the KP operator

$$L_{KP} = \partial + \sum_{l=1}^{\infty} w_l \partial^{-l}.$$

In general, w_l are unrestricted coordinates, while in the realizations (4.9) they are precise functions of the fields a_l and S_l and their derivatives. But that is not all, for one can obtain new

integrable hierarchies *via hamiltonian reduction*. Each integrable hierarchy characterizes a different model. In the case of a reduced hierarchy, we call the corresponding model *a reduced model*. These reduced models will be shown to essentially coincide with the constrained ones.

The results can be synthesized as follows.

Summary. Starting from the Lax operator (4.9) with given m one can show that:
(1) there are $m + 1$ distinct differential integrable hierarchies which are obtained by suppressing successively the fields S_l with the Dirac procedure;
(2) of each such hierarchy it is possible to write down the relevant Lax operator;
(3) at the end of this cascade procedure we find the $(m + 1)$th KdV hierarchy.

Therefore *for every $p = m + 1$ we have p systems or hierarchies, denoted henceforth with the symbol S_p^l, where l counts the number of non-vanishing S fields, $0 \leq l \leq m$. In particular the case $l = m$ corresponds to the $2m$-field representation of the KP hierarchy, while $l = 0$ corresponds to the p-KdV hierarchy.*

The above is general and holds for the more complex systems with $m > 2$. The general case was treated in [26] (see also [24]).

4.1. Examples: The KdV and Boussinesq hierarchies

The simplest example of L, (4.9), corresponds to $m = 1$. It gives rise to the NLS hierarchy

$$L = \partial + a_1 \frac{1}{\partial - S_1}. \tag{4.11}$$

If we impose the constraint $S_1 = 0$, the second Poisson structure can be reduced via the Dirac procedure and leads to a classical version of the Virasoro algebra. The corresponding integrable hierarchy is the KdV hierarchy. Later on we will need the recursion relations for the flows of this hierarchy. They are introduced as follows. Let

$$F_{2k}(x) = \frac{\delta H_{2k}}{\delta a(x)}, \tag{4.12}$$

where H_{2k} are the Hamiltonians, whose explicit form can be derived from the Lax operator, [24,26], and $a \equiv a_1$. Then, imposing the compatibility between the two Poisson brackets, characteristic of the hierarchy, we find the recursion relation for the flows

$$\frac{\partial a}{\partial t_{2k+1}} = F'_{2k+2} = D_F F_{2k}, \qquad D_F = \partial^3 + 4a\partial + 2a' \tag{4.13}$$

with $F_2 = a$.

The simplest integrable system that appears in matrix models after the NLS system is the four-field representation of the KP hierarchy ($m = 2$). It naturally leads, via reduction, to the Boussinesq hierarchy. Let us describe the latter as concisely as possible. The Boussinesq system is described by two fields a_1 and a_2. The Lax operator is

$$L_B = \partial^3 + a_1 \partial + a_2. \tag{4.14}$$

The second Poisson structure is nothing but the classical W_3 algebra. The second flow equations are

$$\frac{\partial}{\partial t_2} a_1 = 2a_2' - a_1'', \qquad \frac{\partial}{\partial t_2} a_2 = a_2'' - \tfrac{2}{3}(a_1 a_1' + a_1'''). \tag{4.15}$$

This is known as the Boussinesq equation (in parametric form) and it is the first of an integrable hierarchy of equations (the Boussinesq or 3-KdV or $\mathcal{S}_3^0$ hierarchy).

Like in the KdV case, we give the recursion relations that allow us to calculate all the flows. Let us define

$$F_r(x) = \frac{\delta H_r}{\delta a_1(x)}, \qquad G_r(x) = \frac{\delta H_r}{\delta a_2(x)}, \qquad r \neq 3n.$$

Then imposing the compatibility of the two relevant Poisson brackets, we find the following recursion relations:

$$\frac{\partial a_1}{\partial t_r} = 3G_{r+3}' = D_{GG} G_r + D_{GF} F_r, \tag{4.16}$$

$$\frac{\partial a_2}{\partial t_r} = 3F_{r+3}' = D_{FG} G_r + D_{FF} F_r \tag{4.17}$$

with $F_1 = 1$, $G_1 = 0$ and $F_2 = 0$, $G_2 = 1$. The differential operators are:

$$D_{GG} = 3a_2 \partial + 2a_2' - a_1 \partial^2 - 2a_1' \partial - a_1'' - \partial^4, \tag{4.18a}$$

$$D_{GF} = 2\partial^3 + 2a_1 \partial + a_1', \tag{4.18b}$$

$$D_{FG} = 2a_2' \partial + a_2'' - \tfrac{2}{3}(a_1 a_1' + a_1^2 \partial + 2a_1 \partial^3 + a_1''' + 3a_1'' \partial + 3a_1' \partial^2 + \partial^5), \tag{4.18c}$$

$$D_{FF} = \partial^4 + a_1 \partial^2 + a_2' + 3a_2 \partial. \tag{4.18d}$$

5. Correlators in reduced models

In this section we show that, starting from the p-KdV or $\mathcal{S}_p^0$ hierarchy and borrowing some information from matrix models, we can define models, i.e. we can define (and compute) a full set of correlators – which turn out to essentially coincide with the constrained models we will meet later on. Since the construction, however, in this section is somewhat heuristic, and, in particular, it does not permit us to carefully fix all the normalizations, we prefer to distinguish these models from the constrained two-matrix models of the following section: as they are based on reduced hierarchies, we refer to them as *reduced models*. We call $\widehat{\mathcal{M}}_p$ the reduced model based on the p-KdV hierarchy. In presenting the reduced models before the constrained two-matrix models, which are the true objectives of our research, we follow a historical and, hopefully, pedagogical order.

The essential definition of the reduced models is as follows: we define the correlators by identifying the field a_1 with the two-point function $\langle \tau_1 \tau_1 \rangle$, i.e. we borrow from the matrix models (Eq. (2.23)), and differentiate (or integrate) a_1 with respect to the couplings, as necessary. Moreover, we only consider the dependence on the $t_k \equiv t_{1,k}$ couplings and

disregard the remaining ones. At this point the flows of the relevant hierarchy allow us to calculate the correlators, at least up to some constants – we are going to see some examples later on. The reason for this is as follows. A part of the information concerning the coupling conditions is in fact stored in the differential system of the model: the Lax operator inherits the information contained in the second equation of (2.13) via the number of non-vanishing diagonal lines of the original $Q(1)$ matrix. Therefore, it is not surprising that the flow equations are almost enough to determine the CFs. However, not all the information concerning the coupling conditions is contained in the differential hierarchy which characterizes the model, the first equation of (2.13) is not, and this is reflected in the undetermined constants that appear when we try to calculate the correlators.

Let us see this point in detail in an explicit example.

5.1. The KdV hierarchy and the associated $\widehat{\mathcal{M}}_2$ model

We showed in Section 4 that we are allowed to impose the constraint $S = 0$ on the NLS system while preserving integrability. In other words there is a consistent subsystem of the NLS system, of which we can easily compute the flows, (4.13). These are the KdV flows. We recall that only the odd flows survive the reduction. Therefore the t_{2n} are disregarded. It is therefore natural to forget $t_0 \equiv N$ as well.

To start with let us define the critical points for this model:

The kth critical point of the $\widehat{\mathcal{M}}_2$ model are defined by $2(2k + 1)t_{2k+1} = -1$ and $t_l = 0$ for $l \neq 1, 2k + 1$.

For the origin of this terminology and further properties of critical points in matrix models see [22]. We will see next that the above critical point corresponds to a two-matrix model with a V_1 potential of order $2k + 1$ and a V_2 potential of cubic order. Let us notice (1) that the correlators of $\widehat{\mathcal{M}}_2$ at the various critical points are functions of t_1 alone; (2) that in order to preserve the homogeneity properties at the kth critical point we have to set $y = y_1(2k + 1)$ in (2.29).

In the following we study the first critical point, $k = 1$. On the basis of the assignments of Section 2.3 we have $[a] = [t_1] = 2y_1$, in genus 0. Therefore it must be: $a \sim t_1$. The proportionality constant can be absorbed with a rescaling (provided it is non-vanishing, which is the case as we shall see). So we start from the position

$$a = t_1. \tag{5.1}$$

Then we have:

Lemma 5.1. *As a consequence of* (5.1), *the functions F_{2n} relevant to the KdV hierarchy are given by*

$$F_{2n} = \sum_{h=0}^{\infty} a(n, h)t^{n-3h}, \quad n \geq 3h, \tag{5.2}$$

where

$$a(n, h) = \frac{2^{n-1}}{12^h h!} \frac{(2n-1)!!}{(n-3h)!},\tag{5.3}$$

and $r!!$ is the 2×2 factorial, i.e. $r!! = r(r-2)\ldots$ up to 1 or 2.

Proof. We insert the expression (5.2) into (4.13) and obtain the recursion relation

$$a(n+1, h) = (n-3h+2)(n-3h+3)a(n, h-1) + 2\frac{2n-6h+1}{n-3h+1}a(n, h)\tag{5.4}$$

for the coefficients $a(n, h)$. One can immediately verify that (5.3) is a solution of (5.4), but it is not unique. While integrating (5.4) one has to specify what $b_h \equiv a(3h, h)$ are $\forall h$. The latter are the coefficients of $(a')^{2h}$ in F_{6h} and satisfy the recurrence relation

$$3hb_h = 2(6h-1)(6h-3)(6h-5)b_{h-1}, \qquad b_1 = 5.$$

One immediately gets (5.3). This ends the proof of the lemma.   □

Proposition 5.2. *The exact one-point correlators of the $\widehat{\mathcal{M}}_2$ model at the first critical point are*

$$\langle \tau_{2n-1} \rangle = \sum_{h=0}^{\infty} \langle \tau_{2n-1} \rangle_h = \sum_{h=0}^{\infty} \frac{2^{n-1}}{12^h h!} \frac{(2n-1)!!}{(n-3h+1)!} t^{n-3h+1}, \quad n \geq 3h-1,\tag{5.5}$$

where the genus expansion is explicitly exhibited.

Proof (partial). We have simply to recall that $F_{2n} = \langle \tau_{2n-1}\tau_1 \rangle$ and integrate over t_1. We obtain (5.5) for $n \geq 3h$. The values of $\langle \tau_{6h-3} \rangle_h$ (i.e. $n = 3h-1$), which are obtained by simple extension of this result, are also correct, but strictly speaking they do not follow from the previous argument: they are pure integration constants and cannot be obtained from the flows alone. We will be able to completely justify Eq. (5.5) only by means of the W constraints. It is in fact the W constraints that completely determine such constants.

5.2. W constraints of the $\widehat{\mathcal{M}}_2$ model

Some information concerning the coupling conditions is not contained in the differential KdV hierarchy. In order to retrieve it we have to use the W constraints. The point is that they cannot be the same as in the unreduced models, since the hierarchy underlying the model has changed, and we recall once again that the W constraints are based on the flow equations. Therefore we have to reconstruct *effective* W constraints on the basis of the reduced hierarchy. Let us argue as follows. In reduced models we are interested in solutions that do not depend on the second sector (i.e. on $t_{2,k}$). If we look at Eq. (2.25), we see that such solutions should therefore satisfy the W constraints of the form

$$\mathcal{L}_n^{[r]}(1)Z_N = 0, \quad r \geq 1; \quad n \geq -r.\tag{5.6}$$

Consequently, we look for W constraints of this type, with generators belonging to a W algebra, which are however compatible with the KdV hierarchy. It is easy to see that the *universal* generators in (2.25), (see [15]), do not satisfy the KdV flows. We find instead:

Proposition 5.3. *The effective W constraints for $\widehat{\mathcal{M}}_2$ take the form*

$$L_n \sqrt{Z} = 0, \quad n \geq -1, \tag{5.7}$$

where:

$$L_{-1} = \sum_{k=1}^{\infty} \left(k + \frac{1}{2}\right) t_{2k+1} \frac{\partial}{\partial t_{2k-1}} + \frac{t_1^2}{16},$$

$$L_0 = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) t_{2k+1} \frac{\partial}{\partial t_{2k+1}} + \frac{1}{16}, \tag{5.8}$$

$$L_n = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) t_{2k+1} \frac{\partial}{\partial t_{2k+2n+1}} + \sum_{k=0}^{n-1} \frac{\partial^2}{\partial t_{2k+1} \partial t_{2n-2k-1}}, \quad n > 0.$$

These generators satisfy the commutation relations of the Virasoro algebra.

Proof. Let us prove first that (5.7) are in agreement with the KdV flows. To this end we differentiate (5.7) with $n > 0$ with respect to t_1. Using the definition of F_{2k}, we can write (remember the notations introduced after Eq. (4.1))

$$\sum_{k=0}^{\infty} (2k+1) t_{2k+1} F_{2k+2n+2} + \partial^{-1} F_{2n+2}$$

$$+ \sum_{k=0}^{n-1} \left(2 \frac{\partial}{\partial t_{2k+1}} F_{2n-2k} + F_{2k+2} \partial^{-1} F_{2n-2k} + \partial^{-1} F_{2k+2} F_{2n-2k}\right) = 0.$$

Here ∂^{-1} is understood in the sense of the pseudodifferential calculus and denotes indefinite integration (see below for further specifications). Now we apply to it the recursion operator D_F. What we obtain, by using Eq. (4.13), is nothing but the constraint $L_{n+1}\sqrt{Z} = 0$ differentiated twice with respect to x. To see this we have to apply the remarkable formula

$$F_{2n+4} = F_{2n+2}'' + 3F_2 F_{2n+2}$$

$$+ \sum_{k=0}^{n-1} (2F_{2k+2} F_{2n-2k}'' - F_{2k+2}' F_{2n-2k}' + 4F_2 F_{2k+2} F_{2n-2k} - F_{2k+2} F_{2n-2k+2}),$$

which can be obtained once again from the recursion relation (4.13). As for the cases $n = 0$ and $n = -1$, which have not been included in the above argument, they can be explicitly verified.

What we have done so far amounts to saying that starting from $L_{-1}\sqrt{Z} = 0$ and successively applying the operator $\mathcal{O} = \partial^{-2} D_F \partial$, we obtain all the $L_n \sqrt{Z} = 0$. Here we have to exercise some care with the double integration ∂^{-2}. ∂^{-1} represents an indefinite integration which preserves the homogeneity properties. This is a perfectly well-defined operation unless the output of it has degree 0. In such a case a numerical integration constant may appear. Now, in $Z^{-1/2} L_n Z^{1/2}$ there appear contributions of degree $2y(1-h) + 2ny_1$, with $n = 0, 1, 2, \ldots$ So, since y and y_1 are generic numbers, the only dangerous case (in the

above sense) is when $h = 1, n = 0$. In other words when we pass from $L_{-1}\sqrt{Z} = 0$ to $L_0\sqrt{Z} = 0$ by applying $\mathcal{O}$ we are not guaranteed that the appropriate constant is given by the $\frac{1}{16}$ present in L_0. However, at this point we make the request that $[L_1, L_{-1}] = 2L_0$, and this unambiguously fixes such constant. It remains for us to justify $L_{-1}\sqrt{Z} = 0$ (which is often referred to as the string equation) or rather the term $\sim t_1^2$ in L_{-1}. From the degree analysis one sees that the only possible polynomial of the couplings one can write is t_1^2. Its coefficient is determined by the requirement that, applying the recursion device to L_{-1}, one gets L_0.

Finally, we do not look for higher tensor constraints since Eq. (5.7) is enough to determine everything.

On the basis of Eq. (5.7) one can complete the proof of Proposition 5.2.   □

5.3. The Boussinesq hierarchy and the associated $\widehat{\mathcal{M}}_3$ model

The 3-KdV or Boussinesq hierarchy was introduced in Section 4.1 as a reduced hierarchy. The corresponding model is denoted $\widehat{\mathcal{M}}_3$. It is described by two fields a_1 and a_2 and is specified by the Lax operator

$$L = \partial^3 + a_1\partial + a_2. \tag{5.9}$$

In the Boussinesq hierarchy the t_{3k} flows with $k = 0, 1, 2, 3 \ldots$ do not appear.

The correlation function interpretation of the fields a_1 and a_2 is given by Eq. (2.23) and the first equation of (4.15):

$$a_1 = \langle \tau_1 \tau_1 \rangle, \qquad 2a_2 = \langle \tau_1 \tau_2 \rangle + \langle \tau_1 \tau_1 \tau_1 \rangle. \tag{5.10}$$

Now we proceed as in the KdV case (but skip many details). The first critical point is defined by $4t_4 = -1$, $t_k = 0$, $k > 4$. This implies in particular that $y = 4y_1$ and that the correlators will be functions of t_1, t_2. Next, the degree analysis shows that $a_1 \sim t_2$ and $a_2 \sim t_1$. An elementary use of the first equation of (4.15) shows that, up to an overall multiplicative normalization constant, we can choose

$$a_1 = 6t_2, \qquad a_2 = 3t_1. \tag{5.11}$$

This will be our choice (as it is consistent with the W constraints and the definition of the critical point). Now it is relatively easy to use the recursion relations of the flow equations to compute the correlation functions.

Proposition 5.4. *The exact one-point correlators of $\widehat{\mathcal{M}}_3$ are*

$$\langle \tau_{3n-2+\epsilon} \rangle = \sum_{h=0}^{\infty} \langle \tau_{3n-2+\epsilon} \rangle_h,$$

$$\langle \tau_{3n-2+\epsilon} \rangle_h$$
$$= \sum_{\substack{j=0 \\ n-j+\epsilon\in 2\mathbb{Z}}}^{n-(8h-2-\epsilon)/3} \frac{1}{48^h h!} \frac{2^{(3n-3j+2+\epsilon)/2-3h}(-1)^{(n-j+\epsilon)/2-h}}{3^{n-j-2+\epsilon}(n-1+\epsilon)!}$$

$$\times \frac{(3n - 3j + 2 + \epsilon - 6h)!!}{(3n - 3j + 2 + \epsilon - 8h)!!}$$

$$\times \frac{(3n - 3 + 2\epsilon)!!! \, (3n - 2 + 2\epsilon)!!!}{(3j)!!! \, ((3n - 3j + \epsilon)/2 - 3h)!!! \, ((3n - 3j + 2 + \epsilon)/2 - 3h)!!!}, \quad (5.12)$$

where $\epsilon = 0, 1$ and $n!!!$ is the 3×3 factorial, i.e. $n!!! = n(n - 3)(n - 6) \ldots$ up to either 1 or 2 or 3. By convention $0!!! = (-1)!!! = 1$, and $1/n!!! = 1/n!! = 0$ for $n \le -2$. As a consequence in the above formula the exponents of t_1 and t_2 are always non-negative.

Proof (sketch). One has to remark first that, while the contributions from two contiguous genera differ by $8y_1$, the recursion operators $D_{GG}, \ldots, D_{FF}$ contain contributions that differ by $4y_1$. It follows that in G_r, F_r there will appear *half-genus* contributions. Therefore, at the first critical point, we have to start from the ansatz (case $\epsilon = 0$)

$$G_{3n+1} = \sum_{s=0}^{\infty} G_{3n+1}^{(s/2)}, \qquad G_{3n+1}^{(s/2)} = \sum_{\substack{j=0 \\ n-j \in 2\mathbb{Z}+1}} \alpha_j(n, s/2) \, t_1^j \, t_2^{(3n-3j-1)/2-2s}, \quad (5.13)$$

$$F_{3n+1} = \sum_{s=0}^{\infty} F_{3n+1}^{(s/2)}, \qquad F_{3n+1}^{(s/2)} = \sum_{\substack{j=0 \\ n-j \in 2\mathbb{Z}}} \beta_j(n, s/2) \, t_1^j \, t_2^{(3n-3j)/2-2s}, \quad (5.14)$$

where s is the half-genus label, i.e. $s = 2h$, and the exponents of t_1 and t_2 are always non-negative. The half-genus contributions must not appear in the correlators, thus we must have the physicality conditions

$$\alpha_j(n, \tfrac{1}{2}s) = 0, \qquad 2\beta_j(n, \tfrac{1}{2}s) = (j + 1)\alpha_{j+1}(n, \tfrac{1}{2}(s - 1)), \qquad s \in 2\mathbb{Z} + 1. \quad (5.15)$$

Plugging (5.14) and (5.13) into (4.16), (4.17), and using (5.15) we find the following recursion relations for the coefficients

$$3j\alpha_j(n + 1, h) = 3(3j - 1)\alpha_{j-1}(n, h) + 12j\beta_j(n, h)$$
$$+ 2j(j + 1)(j + 2)\beta_{j+2}(n, h - 1) \quad (5.16)$$

for $n - j \in 2\mathbb{Z}$, and

$$3j\beta_j(n + 1, h) = 3(3j - 2)\beta_{j-1}(n, h) - 24j\alpha_j(n, h)$$
$$- 5j(j + 1)(j + 2)\alpha_{j+2}(n, h - 1)$$
$$- \tfrac{1}{6}j(j + 1)(j + 2)(j + 3)(j + 4)\alpha_{j+4}(n, h - 2) \quad (5.17)$$

for $n - j \in 2\mathbb{Z} + 1$. These relations can be integrated and give the following result:

$$\alpha_j(n, h) = \frac{1}{48^h h!} \frac{2^{(3n-3j-1)/2-3h}(-1)^{(n-j-1)/2-h}}{3^{n-j-1}(n - 1)!} \frac{(3n - 3j - 1 - 6h)!!}{(3n - 3j - 1 - 8h)!!}$$
$$\times \frac{(3n - 3)!!! \, (3n - 2)!!!}{(3j)!!! \, ((3n - 3j - 1)/2 - 3h)!!! \, ((3n - 3j - 3)/2 - 3h)!!!},$$
$$n - j \in 2\mathbb{Z} + 1;$$

$$\beta_j(n, h) = \frac{1}{48^h h!} \frac{2^{(3n-3j)/2-3h}(-1)^{(n-j)/2-h}}{3^{n-j-1}(n-1)!} \frac{(3n-3j-6h)!!}{(3n-3j-8h)!!}$$

$$\times \frac{(3n-3)!!!\,(3n-2)!!!}{(3j)!!!\,((3n-3j)/2-3h)!!!\,((3n-3j)/2-3h-2)!!!},$$

$$n - j \in 2\mathbb{Z}.$$

Now we recall that

$$\langle \tau_r \tau_1 \rangle_h = 3 G^{(h)}_{r+3}, \qquad \langle \tau_r \tau_2 \rangle_h = 6 F^{(h)}_{r+3}$$

and integrate w.r.t. t_1 and t_2 the first and second expressions, respectively. Comparing the results we find (5.12). Just as in the KdV case we have to treat separately the case when both exponents of t_1 and t_2 in (5.12) vanish. The coefficients given by (5.12) are the correct ones, but they have to be checked by means of the W constraints.

Likewise we can compute $\langle \tau_{3n-1} \rangle$ (case $\epsilon = 1$).   □

The effective W constraints in the case of the $\widehat{\mathcal{M}}_3$ model are found once again by requiring that they be consistent with the Boussinesq flows and that the W generators form a closed algebra.

Proposition 5.5. *The effective W constraints for the $\widehat{\mathcal{M}}_3$ model are*

$$L_n^{[r]} Z^{1/3} = 0, \quad r = 1, 2, \quad n \geq -r, \tag{5.18}$$

where

$$L_n^{[1]} = \frac{1}{3} \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+3n}} + \frac{1}{6} \sum_{\substack{k,l \\ k+l=3n}} \frac{\partial^2}{\partial t_k \partial t_l} + \frac{1}{6} \sum_{\substack{k,l \\ k+l=-3n}} k l t_k t_l + \frac{1}{9} \delta_{n,0} \quad \forall n,$$

$$\sqrt{3} L_n^{[2]} = \frac{1}{9} \sum_{l_1,l_2=1}^{\infty} l_1 l_2 t_{l_1} t_{l_2} \frac{\partial}{\partial t_{l_1+l_2+3n}} + \frac{1}{9} \sum_{\substack{l,k,j \\ l-k-j=-3n}} l t_l \frac{\partial^2}{\partial t_k \partial t_j}$$

$$+ \frac{1}{27} \sum_{\substack{l,k,j \\ l+k+j=3n}} \frac{\partial^3}{\partial t_l \partial t_k \partial t_j} + \frac{1}{27} \sum_{\substack{l,k,j \\ l+k+j=-3n}} l k j t_l t_k t_j \quad \forall n.$$

In these expressions summations are limited to the terms such that no index involved is either negative or multiple of 3. The above two sets of generators form a closed algebra, the W_3 algebra,

$$[L_n^{[1]}, L_m^{[1]}] = (n-m) L_{n+m}^{[1]} + \frac{1}{6}(n^3 - n)\delta_{n+m,0},$$

$$[L_n^{[1]}, L_m^{[2]}] = (2n - m) L_{n+m}^{[2]},$$

$$[L_n^{[2]}, L_m^{[2]}] = -\frac{1}{54}(n-m)((n^2 + m^2 + 4nm) + 3(n+m) + 2) L_{n+m}^{[1]}$$

$$+ \frac{1}{9}(n-m)\Lambda_{n+m} + \frac{1}{810} n(n^2 - 1)(n^2 - 4)\delta_{n+m,0},$$

where

$$\Lambda_n = \sum_{k \leq -1} L_k^{[1]} L_{n-k}^{[1]} + \sum_{k \geq 0} L_{n-k}^{[1]} L_k^{[1]}.$$

This corresponds to the quantum W_3 algebra with central charge 2.

Proof (sketch). One can prove the consistency of the Boussinesq flows with the above constraints in the following way. Call $K_n = Z^{-1/3} L_n^{[1]} Z^{1/3}$. Then, for example, one can check that

$$\frac{1}{2} D_{GF} \frac{\partial K_n}{\partial t_2} + D_{GG} \frac{\partial K_n}{\partial t_1} = \frac{\partial^2 K_{n+1}}{\partial t_1^2}$$

and so on. In fact it is not necessary to prove such equalities for any $n \geq -1$, we simply need to do it for the first few cases, for $K_n = 0$ for $n = -1, 0, 1, 2$ implies $K_n = 0, \forall n \geq -1$ due to the Virasoro algebra structure. The same can be done for the higher-order constraints. Constants and polynomials in the couplings which appear in the generators can be fixed by simply requiring algebraic closure.   □

5.4. Other models

Let us generalize what we have just done for the Boussinesq hierarchy to the $\widehat{\mathcal{M}}_p$ (or pth KdV) models. The general recipe is as follows. One must first of all disregard all the t_k with k a multiple of p; the first critical point is

$$(p + 1)t_{p+1} = b, \qquad t_k = 0, \quad k > p + 1, \tag{5.19}$$

where b is any number. The degree assignment is

$$[t_k] = p + 1 - k, \quad [F^{(h)}] = (2p + 2)(1 - h), \quad [a_1^{(0)}] = 2, \ldots, [a_{p-1}^{(0)}] = p, \tag{5.20}$$

where $a_i^{(0)}$ is the genus 0 part of a_i and we have set, for simplicity $y_1 = 1$. The CFs will be homogeneous functions of $t_1, \ldots, t_{p-1}$, which constitute the small phase space.

In all the cases the method to compute CFs is the same as before. We do not have however to re-do literally the same steps as before. A short cut consists of fixing the form of the fields by means of effective W constraints, which in turn are determined imposing compatibility with the relevant flow equations. Once this is done the CFs can be obtained from the flow equations.

We write down hereafter the W constraints for the general $\widehat{\mathcal{M}}_p$ model. The W constraints are

$$L_n^{[r]} Z^{1/p} = 0, \quad r = 1, \ldots, p - 1, \quad n \geq -r. \tag{5.21}$$

Compact formulae for the above generators can be written down by means of the bosonic formalism. Let us introduce the current

$$J(z) = \sum_{r=1}^{\infty} r t_r z^{r-1} + \frac{1}{p} \sum_{r=1}^{\infty} z^{-r-1} \frac{\partial}{\partial t_r}.$$

Then

$$L_n^{[r]} = \frac{1}{p^{r-1}} \mathrm{Res}_{z=0}(L^{[r]}(z) z^{pn+r}), \qquad L^{[r]}(z) = \frac{1}{r+1} : J(z)^{r+1}.$$

The normal ordering in the last definition is the one between t_r and $\partial/\partial t_r$. The derivative is always supposed to stay at the right. These generators close over a W_p algebra with central charge $p-1$. In particular we have

$$L_n^{[1]} = \frac{1}{p} \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+pn}} + \frac{1}{2p} \sum_{k+l=pn} \frac{\partial^2}{\partial t_k \partial t_l} + \frac{1}{2p} \sum_{k+l=-np} k l t_k t_l + \frac{p^2-1}{24p} \delta_{n,0}.$$

In the above formulae n is any integer, and multiples of p as well as non-positive integers are excluded among the summation indices.

We can extract particular exact formulae as follows: we write down the dispersionless version of the constraints $L_{-r}^{[r]} Z^{1/p} = 0$, with $r = 1, \ldots, p-1$; this equation gives a recursion relation for $\langle \tau_r \rangle$, $r = 1, \ldots, p-1$, in terms of $\langle \tau_l \rangle$ with $l < r$, which can be solved and gives:

$$\langle \tau_r \rangle = \sum_{n=1}^{p-1} (-1)^n \prod_{i=1}^{n-1} \sum_{s_i=1}^{\frac{r_i-1}{2}} b^{-s_i} \binom{r_i}{s_i}$$

$$\times \sum_{\substack{l_1^{(i)}, \ldots, l_{s_i}^{(i)} \\ l_1^{(i)} + \ldots + l_{s_i}^{(i)} + r_i - s_i(p+1) = r_{i+1}}} l_1^{(i)} \ldots l_{s_i}^{(i)} t_{l_1^{(i)}} \ldots t_{l_{s_i}^{(i)}} \sum_{s_n=0}^{r-2} b^{-r+s_n} \binom{r+1}{s_n}$$

$$\times \sum_{\substack{l_1^{(n)}, \ldots, l_{r-s_n+1}^{(n)} \\ l_1^{(n)} + \ldots + l_{r-s_n+1}^{(n)} + s_n(p+1) = pr}} l_1^{(n)} \ldots l_{r-s_n+1}^{(n)} t_{l_1^{(n)}} \ldots t_{l_{r-s_n+1}^{(n)}}. \qquad (5.22)$$

Although this result has been obtained from the genus 0 approximation, *it is an exact result*. In particular, setting $b = -1$, we have

$$\langle \tau_1 \rangle = \frac{p}{2} \sum_{k,l;k+l=p} k l t_k t_l. \qquad (5.23)$$

As a consequence

$$\langle \tau_1 \tau_k \tau_l \rangle = p k l \delta_{l,p-k}, \qquad 1 \leq k, l \leq p-1, \qquad (5.24)$$

which specifies the metric of the corresponding topological field theory [22].

5.5. Higher critical points

In all the previous examples the first critical point has been characterized by a dependence of the basic fields on the couplings specified by homogeneous polynomials with non-negative integer powers. Higher critical points are characterized still by a homogeneous dependence, but with rational and/or negative powers of the couplings.

The procedure to compute correlation functions is always the same: a short cut to arrive at the results is to use simultaneously W constraints and flow equations. With a good deal of perseverance we could probably arrive at exact correlators as in the previous subsections. However, in order to give an idea and for future reference, we think it is enough to present a few partial results.

Let us start with the KdV model. The (Kazakov's) critical points were defined in Section 5.1. At these points the degree assignments (setting $y_1 = 1$) are

$$[t_l] = 2k + 1 - l, \quad [F^{(h)}] = 2(2k + 1)(1 - h), \quad [a^{(h)}] = 2 - (2k + 1)2h.$$

$$(5.25)$$

Indeed, contrary to the first critical point, we have to expect non-vanishing contributions from all genera for the field $a = a^{(0)} + a^{(1)} + a^{(2)} + \cdots$ We find the following results:

$$a^{(0)} = \alpha(k)^{1/k} t_1^{1/k}, \qquad\qquad \alpha(k) = \frac{k!}{2^{k-1}(2k-1)!!},$$

$$a^{(1)} = \beta(k) t_1^{-2}, \qquad\qquad \beta(k) = \frac{1}{24} \frac{k-1}{k},$$

$$a^{(2)} = -\alpha(k)^{-1/k} \gamma(k) t_1^{-(4k+1)/k},$$

where

$$\gamma(k) = \frac{1}{720k^3} (k-1)(32k^3 - 72k^2 + 177k - 77).$$

Knowing these formulae one can calculate the correlators in genus 0, 1 and 2. The expression for $a^{(1)}$ is also in Ref. [29].

Higher critical points of more complicated models can be found in [22].

The models $\widehat{\mathcal{M}}_p$ have a topological field theory interpretation. The corresponding topological field theories are easily identified with those of the A series in the ADE classification [22]. The latter are known to be based on p-KdV hierarchies. Therefore, what we have achieved in this section is a new presentation of this old subject, in fact a very powerful presentation since it has allowed us to calculate new all-genus expressions for the correlators. However, if we look at this section not from the point of view of topological field theories but from the matrix model point of view we cannot yet be satisfied. Although we have used the reduced integrable hierarchies obtained in Section 4 from two-matrix models, and we have used other matrix model inputs, a direct connection between two-matrix models and the results found in this section, although very plausible, has not yet been established: in particular some normalization constants have been arbitrarily fixed and the identification $a_1 \equiv \langle \tau_1 \tau_1 \rangle$ deserves a safer ground.

The purpose of the next section is to provide such connection.

6. Constrained two-matrix models

In this section we want to introduce and analyze constrained two-matrix models that are characterized by p-KdV hierarchies. We are going to study in particular detail the ones based on KdV and Boussinesq hierarchy. In the previous two sections we learned that, in order to end up with the pth KdV hierarchy, we have to impose the constraints $S_i = 0$, which amounts in the Toda lattice formalism to impose that the diagonal elements of the matrix $Q(1)$ vanish identically. We will just impose this constraint on the model $\mathcal{M}_{p_1, p_2}$, and call the two-matrix models so obtained $\mathcal{M}_{p_1, p_2}^{(0)}$. The main result of this section is that the reduced models $\widehat{\mathcal{M}}_p$ can be embedded in the *renormalized* $\mathcal{M}_{p+1, p}^{(0)}$.

A remarkable aspect of our derivation is that *we obtain all the results via the flows of the Toda lattice hierarchy and we never abandon the framework of two-matrix models.* A by-product of this section is a new way to derive p-KdV hierarchies from Toda lattice hierarchies.

6.1. The constrained two-matrix model $\mathcal{M}_{3,2}^{(0)}$ and the KdV hierarchy

The simplest interesting model is $\mathcal{M}_{3,2}^{(0)}$. Let us start from the coupling conditions of the model $\mathcal{M}_{3,2}$ of Section 3.2, and impose the condition $a_0(n) = 0$.

6.1.1. The coupling conditions in $\mathcal{M}_{3,2}^{(0)}$
Remember that KdV, at the first critical point, is expected at $t_2 = 0$ and $6t_3 = -1$. Imposing this and

$$a_0(n) = 0 \tag{6.1}$$

Eqs. (3.8) become, in genus 0,

$$
\begin{aligned}
2cb_2 - R^2 &= 0, & b_1 &= 0, & a_1 &= t_1 - (cs_1/2s_2), \\
R &= -n/c, & b_0(n) &= -s_1/2s_2, & 2s_2R + ca_1 &= 0.
\end{aligned}
\tag{6.2}
$$

Setting for simplicity $s_1 = 0$, one gets in particular

$$a_1 = t_1. \tag{6.3}$$

The importance of (6.3) is that, from the constrained coupling conditions, we have obtained $a_1 = t_1$, a result which was postulated in Section 5.1 on the basis of plausibility arguments.

Remark 6.1. The particular values chosen for s_1, t_2, t_3 are not important. Other choices would not change qualitatively the results, but only renormalize them either additively (s_1) or multiplicatively (t_3, t_2).

Remark 6.2. The last equation of (6.2) implies

$$t_1 = (2s_2/c^2)n. \tag{6.4}$$

Conditions (6.3) and (6.4) have to be imposed as a last condition on the correlators after all the calculations have been carried out (see below), therefore the fact that $t_1 \sim n$ does not interfere with differentiating or integrating with respect to n. The rather mysterious condition (6.4) may have an interesting topological field theory interpretation, see [23].

Remark 6.3. We stress that it is irrelevant whether $a_1 = t_1$ is the unique solution of the coupling conditions and that it be found in genus 0. What is important is that we be able to impose it (at every genus) without breaking integrability. This is in fact what we are going to show next.

6.2. KdV flows from Toda flows

We show next that we can extract the KdV flows directly from the Toda lattice hierarchy. We recall that in $\mathcal{M}_{3,2}$

$$Q(1) = I_+ + \sum_i (a_0(i)E_{i,i} + a_1(i)E_{i,i-1}).$$

Setting $Q \equiv Q(1)$, we have:

Lemma 6.4. In $\mathcal{M}_{3,2}$ the formulae obtained from the Toda lattice hierarchy and from Eq. (2.17)

$$\langle \tau_1 \tau_k \rangle = \mathrm{Tr}([Q_+, Q^k]) \tag{6.5}$$

gives rise to the flows of the NLS hierarchy.

Proof. We notice that, due to Eq. (2.23), we have

$$\frac{\partial a_1}{\partial t_k} = \frac{\partial}{\partial t_1} \langle \tau_1 \tau_k \rangle.$$

This is the first ingredient. The second ingredient are the first flows (2.24a), which, in the $\mathcal{M}_{3,2}$ case, takes the form

$$S_1' = (1 - D_0^{-1})a_1, \qquad a_1' = a_1(D_0 - 1)S_1 \tag{6.6}$$

(remember that we defined $S_1(n) = a_0(n - 1)$). Here we have introduced a new notation, which turns out to be very convenient in this kind of trade. For any discrete function f_N we define

$$(D_0 f)_N = f_{N+1}.$$

We will also use the notation e^{∂_0} instead of D_0, with the following difference: D_0 applies to the nearest right neighbour, while e^{∂_0} is meant to act on whatever is on its right.

Using this notation we can write Q as follows:

$$Q = e^{\partial_0} + D_0 S_1 + a_1 e^{-\partial_0}. \tag{6.7}$$

We also remark that the sum, $\sum_{n=0}^{N-1}$, in Tr is exactly the inverse of the operation $D_0 - 1$. Now it remains for us to evaluate the right-hand side of (6.5). Hereon we give an example (a more complete proof will be provided elsewhere):

$$\begin{aligned}
\frac{\partial a_1}{\partial t_3} &= \frac{\partial}{\partial t_1} \text{Tr}([Q_+, Q^3]) \\
&= \frac{\partial}{\partial t_1} (D_0 a_1 a_1 + a_1 D_0^{-1} a_1 + a_1 a_1 + (D_0 S_1)^2 a_1 + S_1^2 a_1 + a_1 S_1 D_0 S_1) \\
&= (a_1'' + 3a_1^2 + 3S_1^2 a_1 + 3S_1 a_1')',
\end{aligned}$$

which is exactly the third NLS flow for a_1. In order to find the flows for S_1 one has simply to differentiate

$$S_1 = \partial^{-1}(1 - D_0^{-1})a_1$$

and use again the first flows. $\qquad\square$

Using this lemma, it is now easy to conclude our argument. We pass to the $\mathcal{M}_{3,2}^{(0)}$ model setting $S_1 = 0$. The odd flows of a_1 reduce to the KdV flows in exactly the same form given by the recursion relations (4.13). Not only are these relations compatible with the W constraints of Section 5.2, but also the result $a_1 = t_1$, obtained from the coupling conditions, is, and coincides with our assumption (5.1). *We conclude that the model $\widetilde{\mathcal{M}}_2$ is embedded in the constrained two-matrix model $\mathcal{M}_{3,2}^{(0)}$, and that all the results obtained in Sections 5.1 and 5.2 hold true for the latter.*

Remark 6.5. The second sector of $\mathcal{M}_{3,2}^{(0)}$, i.e. the dependence on s_k, as well as the dependence on the bilinear coupling c, does not play a role in the above arguments. They can at most renormalize the final results (as noticed above). In other words, the second sector is a spectator. Whether and how it is possible to compute correlators of the second sector, i.e. correlators of σ_k, or mixed ones, is a question that we leave open here.

6.3. The kth KdV critical point and $\mathcal{M}_{2k+1,2}^{(0)}$

To confirm the result just obtained let us look at the kth KdV critical point (Section 5.5). This critical point turns out to be embedded in the model $\mathcal{M}_{2k+1,2}^{(0)}$. The relevant potentials in this case are

$$V_1(\lambda_1) = t_{2k+1}\lambda_1^{2k+1} + t_1\lambda_1, \qquad V_2(\lambda_2) = s_2\lambda_2^2 + s_1\lambda_2,$$

which entail the following coupling conditions:

$$\begin{aligned}
P^\circ(1) + (2k+1)t_{2k+1}Q(1)^{2k} + t_1 + cQ(2) &= 0, \\
\bar{P}^\circ(2) + 2s_2 Q(2) + s_1 + cQ(1) &= 0.
\end{aligned}$$

The relevant equations one gets in genus 0 are:

$$(2k+1)t_{2k+1}\binom{2k}{k}a_1^k + cb_0 + t_1 = 0,$$

$$2s_2b_0 + s_1 = 0, \qquad 2s_2R + ca_1 = 0, \qquad 2s_2b_1 + cR + n = 0.$$

One finds

$$b_0 = -\frac{s_1}{2s_2}, \qquad a_1 = \left(-\frac{1}{(2k+1)t_{2k+1}}\binom{2k}{k}^{-1}\left(t_1 - \frac{s_1}{2s_2}\right)\right)^{1/k}.$$

Here, either we make the replacement $t_1 \to \tilde{t}_1 = t_1 - s_1/2s_2$ or simply set $s_1 = 0$. Moreover we set (kth critical point) $2(2k+1)t_{2k+1} = -1$. Then

$$a_1 = \left(\frac{k!}{2^{k-1}(2k-1)!!}t_1\right)^{1/k}. \tag{6.8}$$

This result coincides exactly with the analogous formula in Section 5.5. Substituting this in the flows we can calculate the KdV correlators for any critical point in genus 0.

6.4. The renormalized two-matrix model $\mathcal{M}_{4,3}^{(0)}$ and the Boussinesq hierarchy

We now want to repeat the same for the Boussinesq hierarchy. We have to start from the $\mathcal{M}_{4,3}^{(0)}$, but one soon realizes that things are not as simple as in the previous example. In fact an analogue of Lemma 6.4 holds, but setting $a_0 = 0$ in the resulting flows does not lead to the Boussinesq flows. A significant change of strategy is necessary. But let us start once again from the coupling conditions for $\mathcal{M}_{4,3}$.

6.4.1. The coupling conditions in $\mathcal{M}_{4,3}^{(0)}$

The coupling conditions for the two-matrix model $\mathcal{M}_{4,3}$ are

$$P^\circ(1) + 4t_4Q(1)^3 + 3t_3Q(1)^2 + 2t_2Q(1) + t_1 + cQ(2) = 0,$$
$$\bar{P}^\circ(2) + 3s_3Q(2)^2 + 2s_2Q(2) + s_1 + cQ(1) = 0. \tag{6.9}$$

These can be expressed in terms of the fields $a_0, a_1, a_2, b_0, b_1, b_2, b_3$ and R. We expect to find the Boussinesq hierarchy for $a_0 = 0$ and $t_3 = 0$. The coupling conditions in genus 0 then become

$$cb_3 = -4t_4R^3, \qquad cb_2 = 0, \qquad cb_1 = -2t_2R - 12t_4a_1R,$$

$$cb_0 = -t_1 - 12t_4a_2, \qquad n + 12t_4a_1^2 + 2t_2a_1 + cR = 0, \qquad 3s_3R^2 + ca_2 = 0,$$

$$6s_3b_0R + 2s_2R + ca_1 = 0, \qquad s_3(2b_1 + b_0^2) + 2s_2b_0 + s_1 = 0,$$

$$6s_3b_0b_1 + 2s_2b_1 + cR + n = 0.$$

For simplicity we choose $b_1 = 0$ and $s_1 = 0$, which leads to

$$a_1 = -\frac{t_2}{6t_4}, \qquad a_2 = -\frac{t_1}{12t_4}, \qquad R = -\frac{n}{c}. \tag{6.10}$$

After determining $b_0 = 0$ and b_3 we are left with two conditions of the type (6.4) among the couplings, which are irrelevant for the following developments. Different choices of b_1 and s_1 would simply imply additive redefinitions (renormalizations) of t_1 and t_2, therefore we ignore them.

It is important that up to a global rescaling (see below) we have found the same results that were assumed under (very) plausible arguments in Section 5.3, see Eq. (5.11).

6.4.2. Boussinesq flows from Toda flows

As we have anticipated above, in the $\mathcal{M}_{4,3}$ there is an analogue of Lemma 6.4. In particular, from the Toda flows we can obtain the flows of the four-field KP hierarchy. However this is irrelevant to our problem, since, setting $S_1 = S_2 = 0$ does not lead to the Boussinesq hierarchy (or to any integrable hierarchy, for that matter). The reason is well known: the above conditions oblige the system to flow outside the manifolds of the flow equations. To preserve integrability we have to introduce in the original hierarchy a (presumably infinite) set of corrections. In field theory language we can say that the constraint $a_0 = 0$ can be imposed without spoiling integrability only at the price of introducing a (presumably infinite) set of counter-terms. The very important point is that this set of counter-terms can be exactly computed, after which the resulting model, referred to henceforth as the $\mathcal{M}_{4,3}^{(0,r)}$ model, will accomodate the Boussinesq hierarchy.

The recipe to obtain the result is as follows:

(1) Define the general matrix

$$\widehat{Q} = e^{\partial_0} + \sum_{i=1}^{\infty} \hat{a}_i e^{-i\partial_0}. \tag{6.11}$$

(2) Assume as first flows

$$D_0 \hat{a}_1 = \hat{a}_1, \qquad D_0 \hat{a}_i = \hat{a}_i + \hat{a}'_{i-1}, \quad i \geq 2 \tag{6.12}$$

(these are the usual first flows in which we have set $\hat{a}_0 = 0$).

(3) Now impose

$$\widehat{Q}^3 = e^{3\partial_0} + 3a_1 e^{\partial_0} + 3a_2. \tag{6.13}$$

This equation and (6.12) completely determine $\hat{a}_i$ in terms of a_1, a_2,

$$\hat{a}_1 = \bar{a}_1 \equiv a_1, \qquad \hat{a}_2 = \bar{a}_2 \equiv a_2 - a'_1, \qquad \hat{a}_3 = \bar{a}_3 \equiv -a'_2 - a_1^2 + \tfrac{2}{3}a''_1,$$
$$\hat{a}_4 = \bar{a}_4 \equiv \tfrac{2}{3}a''_2 - \tfrac{1}{3}a'''_1 + 4a_1 a'_1 - 2a_1 a_2$$

and so on. $\widehat{Q}$ is now to replace the matrix $Q(1)$ of the model $\mathcal{M}_{4,3}$ with $a_0 = 0$. It contains the right counter-terms to generate the Boussinesq hierarchy.

(4) To obtain the above, we use the following lemma.

Lemma 6.6. *Replace $Q(1)$ with $\widehat{Q}$ in the Toda lattice hierarchy formulae and evaluate them at $\hat{a}_i = \bar{a}_i$*

$$\frac{\partial a_1}{\partial t_k} = \frac{\partial}{\partial t_1} \text{Tr}([\widehat{Q}_+, \widehat{Q}^k])\Big|_{\hat{a}_i = \bar{a}_i}, \qquad k \neq 3n \tag{6.14}$$

and the like. Then these formulae provide a realization of the Boussinesq hierarchy.

Proof. We limit ourselves to a few examples (a more complete proof will be given elsewhere):

$$\frac{\partial a_1}{\partial t_2} = \frac{\partial}{\partial t_1} \text{Tr}([\widehat{Q}_+, \widehat{Q}^2])\Big|_{\hat{a}_i = \bar{a}_i} = (D_0 \hat{a}_2 + \hat{a}_2)'|_{\hat{a}_i = \bar{a}_i} = 2a_2' - a_1'',$$

$$\frac{\partial a_1}{\partial t_4} = \frac{\partial}{\partial t_1} \text{Tr}([\widehat{Q}_+, \widehat{Q}^4])\Big|_{\hat{a}_i = \bar{a}_i} = \tfrac{2}{3}a_2'' - \tfrac{1}{3}a_1''' - 2a_1 a_1' + 4a_1 a_2,$$

$$\partial^{-1} \frac{\partial a_2}{\partial t_2} = \tfrac{1}{2}\text{Tr}([\widehat{Q}_+^2, \widehat{Q}^2])\Big|_{\hat{a}_i = \bar{a}_i} + \frac{1}{2}\frac{\partial a_1}{\partial t_2} = a_2' - a_1^2 - \tfrac{2}{3}a_1''$$

and so on. These are the Boussinesq flows, even though not in exactly the same form as in Sections 4.1 and 5.3. In fact we have to multiply by 3 the a_1, a_2 fields there to obtain the flows here.   □

Remark 6.7. We should have allowed also for a field a_0 in $\tilde{Q}$ and set it to zero at the end of the calculations. This could of course have been done but would not have changed the results. Setting $a_0 = 0$ from the very beginning we have simply anticipated the result and simplified the formulae a lot.

Remark 6.8. It is not surprising that we have found some disagreements in the normalizations compared here in Sections 4 and 5. We have already pointed out at the end of Section 5 that some identifications made there were likely to be arbitrary as far as the normalizations are concerned.

Up to the normalization problem illustrated in the previous remark we can see that *the flow equations, the coupling conditions* (6.10) *and consequently the W constraints pertinent to the $\widehat{\mathcal{M}}_3$ model can be embedded in the $\mathcal{M}_{4,3}^{(0,r)}$ matrix model.* If we set the critical point at $12t_4 = -1$ in (6.10) and multiply by 3 the fields a_1, a_2 in Sections 4 and 5, we can simply transfer the results obtained there to $\mathcal{M}_{4,3}^{(0,r)}$. To be more precise, we summarize the results concerning the latter as follows:

(i) The Boussinesq flows are given by

$$\frac{\partial a_1}{\partial t_r} = G'_{r+3} = D_{GG} G_r + D_{GF} F_r, \tag{6.15}$$

$$\frac{\partial a_2}{\partial t_r} = F'_{r+3} = D_{FG} G_r + D_{FF} F_r \tag{6.16}$$

with $F_1 = 1$, $G_1 = 0$ and $F_2 = 0$, $G_2 = 1$. The differential operators are:

$$D_{GG} = 3a_2 \partial + 2a_2' - a_1 \partial^2 - 2a_1' \partial - a_1'' - \tfrac{1}{3}\partial^4, \tag{6.17a}$$

$$D_{GF} = \tfrac{2}{3}\partial^3 + 2a_1\partial + a_1', \tag{6.17b}$$

$$D_{FG} = 2a_2'\partial + a_2'' - 2(a_1a_1' + a_1^2\partial + \tfrac{2}{3}a_1\partial^3$$
$$+ \tfrac{1}{3}a_1''' + a_1''\partial + a_1'\partial^2 + \tfrac{1}{9}\partial^5), \tag{6.17c}$$

$$D_{FF} = \tfrac{1}{3}\partial^4 + a_1\partial^2 + a_2' + 3a_2\partial. \tag{6.17d}$$

(ii) The coupling conditions imply, at $12t_4 = -1$,

$$a_1 = 2t_2, \qquad a_2 = t_1. \tag{6.18}$$

(iii) The correlators are the same as in Section 5.5 except for a global factor of 3. In particular the right-hand side of (5.12) must be divided by 3.

(iv) The W constraints appropriate for $\mathcal{M}_{4,3}^{(0,r)}$ are

$$L_n^{[r]} Z = 0, \qquad r = 1, 2, \quad n \geq -r \tag{6.19}$$

the generators being the same as in Proposition 5.5. But in order to reproduce the right correlators we have to compute them at $4t_4 = -1$. In other words the location of the critical point gets renormalized.

7. Conclusion

The procedure introduced above for the Boussinesq hierarchy holds for any p-KdV hierarchy. We can always define suitable coordinates in the matrix $\widehat{Q}$, which, inserted in the Toda lattice flows, generate the p-KdV flows. The recipe is just a generalization of the one given above. This is certainly a remarkable (and new, to our knowledge) result, which deserves further elaboration.

Finally, we can draw the following conclusion: the p-KdV hierarchies are contained in specific *constrained* two-matrix models; once we impose the constraint, the Q matrix of the relevant model has to be suitably redefined (except in the 2-KdV case) to ensure integrability; the counter-terms can be exactly calculated and give rise to "renormalized" coordinates; in turn these coordinates, when substituted in the formulae of the Toda lattice hierarchy, gives rise to the p-KdV flows. The same procedure may well be applicable to extract from two-matrix models other hierarchies such as those studied in [30].

Appendix A

Here are the exact 2 point CFs of the model $\mathcal{M}_{2,2}$. Let us first define the functions

$$\mathcal{F}_{n,m}(\alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) = \sum_{l=0}^{n}\sum_{k=0}^{l/2}\sum_{r=0}^{(l-2k)/2}\sum_{p=0}^{m}\sum_{q=0}^{p/2}\frac{n!\,m!}{2^{k+q}(n-l)!(m-p)!k!q!}$$

$$\times \sum_{j=0}^{\min(l-2k,\,p-2q)}\left(\frac{1}{r!j!(r+k-q)+\tfrac{1}{2}(p-l)!(l-2k-r-j)!(\tfrac{1}{2}(l+p)-k-q-r-j)!}\right.$$

$$-\frac{1}{j!(l-2k-r)!(r-j)!(\frac{1}{2}(l+p)-k-q-r)!(\frac{1}{2}(p-l)-q+k+r-j)!}\Bigg)$$

$$\times(\tfrac{1}{2}(l+p)-k-q-j)!\left(\begin{array}{c} N \\ \frac{1}{2}(l+p)-k-q-j+1 \end{array}\right)$$

$$\times\alpha_2^{(p-l)/2+k+r}\beta_1^{n-l}\beta_2^{m-p}\gamma_1^{k+r}\gamma_2^{(p+l/2)-k-r},$$

where α_2, β_1, β_2, γ_1, γ_2 were defined in Eq. (3.4). Then

$$\langle\tau_n\sigma_m\rangle = \mathrm{Tr}([Q(1)^n, Q(2)^m_-]) = \mathcal{F}_{n,m}(\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2), \tag{A.1}$$

$$\langle\tau_n\tau_m\rangle = \mathrm{Tr}([Q(1)^n, Q(1)^m_-]) = \mathcal{F}_{n,m}(1,\beta_1,\beta_1,\gamma_1,\gamma_1). \tag{A.2}$$

Finally $\langle\sigma_n\sigma_m\rangle$ is obtained from $\langle\tau_n\tau_m\rangle$ with the exchange $t_k \leftrightarrow s_k$ for $k = 1, 2$.

Acknowledgements

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